

# STRUCTURE COEFFICIENTS FOR JACK CHARACTERS: APPROXIMATE FACTORIZATION PROPERTY

PIOTR ŚNIADY

**ABSTRACT.** *Jack characters* are a generalization of the characters of the symmetric groups; a generalization that is related to Jack symmetric functions. We investigate the *structure coefficients* for Jack characters; they are a generalization of the connection coefficients for the symmetric groups. More specifically, we study the *cumulants* which measure the discrepancy between these structure coefficients and the simplistic structure coefficients related to the *disjoint product*. We show that Jack characters satisfy *approximate factorization property*: their cumulants are of very small degree and the character related to a given partition is approximately equal to the product of the characters related to its parts. This result will play a key role in the proof of Gaussianity of fluctuations for a class of random Young diagrams related to the Jack polynomials.

## 0. PROLOGUE

We study algebraic combinatorial problems related to *Jack characters*  $\text{Ch}_\pi$ , which form a natural family (indexed by partition  $\pi$ ) of functions on the set  $\mathbb{Y}$  of Young diagrams. Jack characters can be viewed as a natural deformation of the classical *normalized characters of the symmetric groups*; a deformation that is associated with *Jack symmetric functions*. In order to intrigue the Reader and to keep her motivated we shall present now some selected highlights before getting involved in somewhat lengthy definitions in the Introduction. We also postpone the bibliographic details.

Some Readers may prefer to fast forward directly to Section 0.11 where concrete motivations for our research are discussed.

**0.1. Structure coefficients.** The main problem which we address in current paper is *to understand the structure coefficients of Jack characters*. More specifically, for any partitions  $\pi$  and  $\sigma$  one can uniquely express the

---

2010 *Mathematics Subject Classification*. Primary 05E05; Secondary 05E10 20C30, 05E15,

*Key words and phrases*. Jack polynomials, Jack characters, structure coefficients, approximate factorization of characters.

Version identifier: 3a5fd52 .

pointwise product of the corresponding Jack characters

$$(0.1) \quad \text{Ch}_\pi \text{Ch}_\sigma = \sum_{\mu} g_{\pi,\sigma}^{\mu}(\delta) \text{Ch}_{\mu}$$

in the linear basis of Jack characters. The coefficients  $g_{\pi,\sigma}^{\mu}(\delta) \in \mathbb{Q}[\delta]$  in this expansion are called *structure coefficients*. Each of them is a polynomial in the deformation parameter  $\delta$ , on which Jack characters depend implicitly. The existence of these polynomials is a non-trivial fact [DF16, Theorem 1.4]. For example,

$$\begin{aligned} \text{Ch}_3 \text{Ch}_2 &= 6\delta \text{Ch}_3 + \text{Ch}_{3,2} + 6\text{Ch}_{2,1} + 6\text{Ch}_4, \\ \text{Ch}_3 \text{Ch}_3 &= (6\delta^2 + 3) \text{Ch}_3 + 9\delta \text{Ch}_{2,1} + 18\delta \text{Ch}_4 + 3\text{Ch}_{1,1,1} + \\ &\quad + 9\text{Ch}_{3,1} + 9\text{Ch}_{2,2} + 9\text{Ch}_5 + \text{Ch}_{3,3}. \end{aligned}$$

Exploration of the numerical examples such as the ones above suggests that the structure coefficients for Jack characters might have a rich algebraic and combinatorial structure and encourages to state the following conjecture.

**Conjecture 0.1** (Structure coefficients of Jack characters). *For any partitions  $\pi, \sigma, \mu$  the corresponding structure coefficient*

$$g_{\pi,\sigma}^{\mu}(\delta) \in \mathbb{Q}[\delta]$$

*is a polynomial with non-negative integer coefficients.*

**0.2. Symmetric groups.** As we already mentioned, Jack characters implicitly depend on some deformation parameter. In the special case when  $\delta = 0$  their structure coefficients have a known, classical form which will be discussed in the following.

Let us fix an integer  $n \geq 1$ . For a partition  $\pi = (\pi_1, \dots, \pi_\ell)$  we consider a specific element of the symmetric group algebra  $\mathbb{C}[\mathfrak{S}(n)]$ . This element  $A_{\pi;n}$  is equal (up to some normalization constants) to the indicator of the conjugacy class in the symmetric group  $\mathfrak{S}(n)$ , which consists of permutations with the cycle decomposition given by  $\pi$ ; possibly with some additional fixpoints. Impatient Readers may fast forward to Section 0.3 since now we are going to present some fine details of the definition of  $A_{\pi;n}$ .

We define  $A_{\pi;n}$  as the following formal combination of the permutations [KO94, IK99, Bia03]:

$$A_{\pi;n} = \sum_f (f_{1,1}, f_{1,2}, \dots, f_{1,\pi_1}) \cdots (f_{\ell,1}, f_{\ell,2}, \dots, f_{\ell,\pi_\ell}).$$

The above sum runs over *injective* functions

$$f: \{(i, j) : 1 \leq i \leq \ell, 1 \leq j \leq \pi_i\} \rightarrow \{1, \dots, n\}.$$

In other words, we fill the boxes of the Young diagram  $\pi$  with the elements of the set  $[n] := \{1, \dots, n\}$  in such a way that each number is used at most once; for each filling we interpret the rows as cycles in the cycle decomposition of the permutation.

The definition of the elements  $A_{\pi;n}$  can be further improved by replacing the notion of permutations by *partial permutations* [IK99]. This allows to define some abstract *conjugacy class indicators*  $A_\pi$  which are elements of some suitable inverse limit and thus do not depend on  $n$ , the rank of the symmetric group  $\mathfrak{S}(n)$ .

**0.3. The case  $\delta = 0$  and the symmetric groups.** One can show the following fact.

**Fact 0.2.** *The specialization  $\delta = 0$  of the polynomials  $g_{\pi,\sigma}^\mu$  gives the structure coefficients for the family  $(A_\pi)$  of the conjugacy class indicators.*

*In other words:*

$$A_\pi A_\sigma = \sum_{\mu} g_{\pi,\sigma}^\mu(0) A_\mu,$$

where the product on the left-hand side is the usual convolution product in the group algebra  $\mathbb{C}[\mathfrak{S}(n)]$  (or, more precisely, the convolution product in the semigroup algebra of partial permutations).

The above fact sheds some light on the structure coefficients  $g_{\pi,\sigma}^\mu \in \mathbb{Q}[\delta]$ . Firstly, since the structure coefficients for the conjugacy classes  $A_\pi$  are non-negative integers with a very well-understood combinatorial interpretation (they are the connection coefficients for the symmetric groups) [IK99], the same can be said about the evaluation  $g_{\pi,\sigma}^\mu(0)$ .

Secondly, the above Fact 0.2 suggests that a plausible explanation of Conjecture 0.1 would be the following one.

**Conjecture 0.3.** *There exists some natural deformation of the symmetric group algebra  $\mathbb{C}[\mathfrak{S}(n)]$  which depends on the deformation parameter  $\delta$  in which the structure coefficients for the hypothetical ‘conjugacy class indicators’ are given by the structure coefficients  $g_{\pi,\sigma}^\mu$  of Jack characters.*

**0.4. Kerov–Lassalle positivity conjecture, simplistic version.** *Free cumulants  $\mathcal{R}_2, \mathcal{R}_3, \dots$  are a certain convenient family of functions on the set of Young diagrams; they provide a description of the macroscopic shape of a given Young diagram.*

In this section it will be more convenient to use a different deformation parameter on which the Jack characters depend implicitly; it is the variable  $\gamma$  given by

$$\gamma = -\delta.$$

Also free cumulants depend implicitly on  $\gamma$ .

It has been proved by Lassalle [Las09a] that for each partition  $\pi$  there exists a unique polynomial, called *Kerov–Lassalle polynomial*, which gives the values of the Jack character  $\text{Ch}_\pi$ . For example, in the simplest case when the partition  $\pi = (k)$  consists of a single part,

$$\text{Ch}_1 = \mathcal{R}_2,$$

$$\text{Ch}_2 = \mathcal{R}_3 + \mathcal{R}_2\gamma,$$

$$\text{Ch}_3 = \mathcal{R}_4 + 3\mathcal{R}_3\gamma + 2\mathcal{R}_2\gamma^2 + \mathcal{R}_2,$$

$$\text{Ch}_4 = \mathcal{R}_5 + 6\mathcal{R}_4\gamma + \mathcal{R}_2^2\gamma + 11\mathcal{R}_3\gamma^2 + 6\mathcal{R}_2\gamma^3 + 5\mathcal{R}_3 + 7\mathcal{R}_2\gamma.$$

Based on such numerical examples, Lassalle [Las09a] formulated the following challenging conjecture.

**Conjecture 0.4** (Kerov–Lassalle positivity conjecture). *For each integer  $k \geq 1$  the polynomial which expresses the Jack character  $\text{Ch}_k$  in terms of the variables  $\gamma, \mathcal{R}_2, \mathcal{R}_3, \dots$  has non-negative integer coefficients.*

Note that in the current paper we use a normalization of Jack characters and free cumulants based on the paper of Dołęga and Féray [DF16] which differs from the normalization from the original paper of Lassalle [Las09a].

**0.5. Covariance for Jack characters.** One can wonder if Conjecture 0.4 would hold true for Kerov–Lassalle polynomials corresponding to more complicated Jack characters  $\text{Ch}_\pi$  in the case when the partition  $\pi$  consists of more than one part. Regretfully, this is not the case, as one can see on the following simple example

$$(0.2) \quad \text{Ch}_{2,2} =$$

$$\mathcal{R}_3^2 + 2\mathcal{R}_3\mathcal{R}_2\gamma + \mathcal{R}_2^2\gamma^2 - 4\mathcal{R}_4 - 2\mathcal{R}_2^2 - 10\mathcal{R}_3\gamma - 6\mathcal{R}_2\gamma^2 - 2\mathcal{R}_2$$

which contains both positive and negative coefficients in the expansion. In other words, Jack characters  $\text{Ch}_\pi$  corresponding to such more complicated partitions  $\pi$  *are not the right quantities to consider* and we should look for a convenient replacement.

It turns out that for partitions  $\pi = (a, b)$  which consist of two parts, instead of the character  $\text{Ch}_{a,b}$  it is more convenient to consider a kind of *covariance*

$$(0.3) \quad \kappa_\bullet(\text{Ch}_a, \text{Ch}_b) := \text{Ch}_{a,b} - \text{Ch}_a \text{Ch}_b.$$

For example, in the case considered in Eq. (0.2) above, the corresponding Kerov–Lassalle polynomial is given by

$$\kappa_\bullet(\text{Ch}_2, \text{Ch}_2) = -[4\mathcal{R}_4 + 2\mathcal{R}_2^2 + 10\mathcal{R}_3\gamma + 6\mathcal{R}_2\gamma^2 + 2\mathcal{R}_2].$$

As one can see in this example, apart from a global change of the sign, the positivity of the coefficients is restored which supports the claim that the covariance  $\kappa_\bullet(\text{Ch}_a, \text{Ch}_b)$  is indeed *the right* quantity.

The remaining question is: how to generalize these covariances (0.3) for even more complicated partitions  $\pi$ ? We shall address this question in the following.

**0.6. Cumulants in classical probability theory.** A problem that is analogous to the one above appears in the probability theory: *how to describe the joint distribution of a family  $(X_i)$  of random variables in the most convenient way?* A simple solution is to use the family of *moments*, which are just the expected values of the products of the form

$$\mathbb{E}X_{i_1} \cdots X_{i_l}.$$

However, the simplest solution is often not the best one, and this is also the case here.

It has been observed that it is much more convenient to pass to some new quantities, called *cumulants* or *semi-invariants* [Hal81, Fis28], which are defined as the coefficients of the expansion of the logarithm of the multidimensional Laplace transform around zero:

$$(0.4) \quad \kappa(X_1, \dots, X_n) = [t_1 \cdots t_n] \log \mathbb{E}e^{t_1 X_1 + \cdots + t_n X_n} = \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E}e^{t_1 X_1 + \cdots + t_n X_n} \right|_{t_1 = \cdots = t_n = 0}.$$

Each such a cumulant is a linear map with respect to each of its arguments.

There are many good reasons for claiming superiority of the cumulants over the moments; we shall discuss very briefly only one of them. Roughly speaking, the convolution of measures corresponds to the product of the Laplace transforms or, in other words, to *the sum of the logarithms of the Laplace transforms*. It follows that the cumulants behave in a very simple way with respect to the convolution, namely *cumulants linearize the convolution*.

Cumulants allow also a combinatorial description: one can show that the definition (0.4) is equivalent to the following system of equations:

$$(0.5) \quad \mathbb{E}(X_1 \cdots X_n) = \sum_{\nu} \prod_{b \in \nu} \kappa(X_i : i \in b)$$

which should hold true for any choice of the random variables  $X_1, \dots, X_n$  for which all their moments are finite. The above sum runs over the set-partitions  $\nu$  of the set  $[n]$  and the product runs over the blocks of the partition

$\nu$  (for missing definitions see Section 1.2). For example, we require that

$$(0.6) \quad \left\{ \begin{array}{l} \mathbb{E}(X_1) = \kappa(X_1), \\ \mathbb{E}(X_1 X_2) = \kappa(X_1, X_2) + \kappa(X_1)\kappa(X_2), \\ \mathbb{E}(X_1 X_2 X_3) = \kappa(X_1, X_2, X_3) + \kappa(X_1)\kappa(X_2, X_3) \\ \quad + \kappa(X_2)\kappa(X_1, X_3) + \kappa(X_3)\kappa(X_1, X_2) \\ \quad + \kappa(X_1)\kappa(X_2)\kappa(X_3), \\ \vdots \end{array} \right.$$

**0.7. Cumulants of Jack characters.** We will use the above ideas from the classical probability theory as a heuristic motivation for some new functions  $\kappa_\bullet(\text{Ch}_a, \dots, \text{Ch}_z)$  on the set of Young diagrams, for arbitrary integers  $a, b, c, \dots \geq 1$ . Roughly speaking, we rewrite the system of equations (0.6) as follows: for the random variables  $X_1 = \text{Ch}_a, X_2 = \text{Ch}_b, \dots$  we take the Jack characters corresponding to partitions with only one part (*single letter*). As the moment on the left-hand side of (0.6) we take the Jack character corresponding to the partition equal to the *concatenation* of the letters. For example,

$$(0.7) \quad \left\{ \begin{array}{l} \text{Ch}_a = \kappa_\bullet(\text{Ch}_a), \\ \text{Ch}_{a,b} = \kappa_\bullet(\text{Ch}_a, \text{Ch}_b) + \kappa_\bullet(\text{Ch}_a)\kappa_\bullet(\text{Ch}_b), \\ \text{Ch}_{a,b,c} = \kappa_\bullet(\text{Ch}_{a,b,c}) + \kappa_\bullet(\text{Ch}_a)\kappa_\bullet(\text{Ch}_b, \text{Ch}_c) \\ \quad + \kappa_\bullet(\text{Ch}_b)\kappa_\bullet(\text{Ch}_a, \text{Ch}_c) + \kappa_\bullet(\text{Ch}_c)\kappa_\bullet(\text{Ch}_a, \text{Ch}_b) \\ \quad + \kappa_\bullet(\text{Ch}_a)\kappa_\bullet(\text{Ch}_b)\kappa_\bullet(\text{Ch}_c), \\ \vdots \end{array} \right.$$

We shall revisit this definition in Section 1.12.

The system of equations (0.7) can be iteratively solved. For example,

$$\left\{ \begin{array}{l} \kappa_\bullet(\text{Ch}_a) = \text{Ch}_a, \\ \kappa_\bullet(\text{Ch}_a, \text{Ch}_b) = \text{Ch}_{a,b} - \text{Ch}_a \text{Ch}_b, \\ \vdots \end{array} \right.$$

in other words the first cumulant  $\kappa_\bullet(\text{Ch}_a)$  coincides with the usual Jack character and the second cumulant  $\kappa_\bullet(\text{Ch}_a, \text{Ch}_b)$  coincides with the covariance (0.3).

These cumulants are in the focus of the current paper. In the following Sections 0.8–0.10 we shall discuss a couple of ways in which the following rather vague claim can be made concrete.

**Vague Claim 0.5.** *The cumulants  $\kappa_\bullet$  have a much nicer structure than Jack characters themselves.*

**0.8. Kerov–Lassalle conjecture for the cumulants.** The first advantage of the cumulants over the characters is related to *positivity*. For example, Kerov–Lassalle polynomial for one of the cumulants is given by

$$(0.8) \quad \kappa_\bullet(\text{Ch}_2, \text{Ch}_2, \text{Ch}_2) = 40\mathcal{R}_5 + 64\mathcal{R}_3\mathcal{R}_2 + 176\mathcal{R}_4\gamma + 96\mathcal{R}_2^2\gamma + 256\mathcal{R}_3\gamma^2 + 80\mathcal{R}_3 + 120\mathcal{R}_2\gamma^3 + 104\mathcal{R}_2\gamma$$

which should be compared to the analogous expression for the corresponding character:

$$(0.9) \quad \begin{aligned} \text{Ch}_{2,2,2} = & \mathcal{R}_3^3 + 3\mathcal{R}_3^2\mathcal{R}_2\gamma - 12\mathcal{R}_4\mathcal{R}_3 + 3\mathcal{R}_3\mathcal{R}_2^2\gamma^2 - 6\mathcal{R}_3\mathcal{R}_2^2 \\ & - 30\mathcal{R}_3^2\gamma - 12\mathcal{R}_4\mathcal{R}_2\gamma + \mathcal{R}_2^3\gamma^3 - 6\mathcal{R}_2^3\gamma + 40\mathcal{R}_5 - 48\mathcal{R}_3\mathcal{R}_2\gamma^2 + \\ & 58\mathcal{R}_3\mathcal{R}_2 + 176\mathcal{R}_4\gamma - 18\mathcal{R}_2^2\gamma^3 + 90\mathcal{R}_2^2\gamma + 256\mathcal{R}_3\gamma^2 + 80\mathcal{R}_3 + \\ & 120\mathcal{R}_2\gamma^3 + 104\mathcal{R}_2\gamma. \end{aligned}$$

The coefficients on the right-hand side of (0.8) are all positive, which is a great advantage from the perspective of an algebraic combinatorialist. This positivity does not hold for (0.9). Based on such numerical evidence, Lassalle [Las09a] stated the following conjecture which is an extension of Conjecture 0.4.

**Conjecture 0.6** (Kerov–Lassalle positivity conjecture). *For all integers  $k_1, \dots, k_\ell \geq 1$  the polynomial which expresses the (signed) cumulant*

$$(-1)^{\ell-1} \kappa_\bullet(\text{Ch}_{k_1}, \dots, \text{Ch}_{k_\ell})$$

*in terms of the variables  $\gamma, \mathcal{R}_2, \mathcal{R}_3, \dots$  has non-negative integer coefficients.*

Clearly, this conjecture gives some support to Claim 0.5.

**0.9. Structure coefficients revisited.** One can show that the product (0.1) is always of the form

$$\text{Ch}_\pi \text{Ch}_\sigma = \text{Ch}_{\pi\sigma} + \dots,$$

where the product  $\pi\sigma$  denotes the *concatenation* of the partitions. In other words, the corresponding structure coefficient

$$g_{\pi,\sigma}^{\pi\sigma} = 1$$

is particularly simple.

It follows that Conjecture 0.1 is equivalent to the following statement: *each (signed) coefficient  $(-1)d_{\pi,\sigma}^\mu \in \mathbb{Q}[\delta]$  in the expansion*

$$\kappa_\bullet(\text{Ch}_\pi, \text{Ch}_\sigma) = \text{Ch}_{\pi\sigma} - \text{Ch}_\pi \text{Ch}_\sigma = \sum_{\mu} d_{\pi,\sigma}^\mu(\delta) \text{Ch}_\mu$$

*is a polynomial in  $\delta$  with non-negative integer coefficients.* The left-hand side is a generalization of the covariance (0.3) to the case when partitions  $\pi$  and  $\sigma$  have more than one part. In particular, Conjecture 0.1 is a special case  $\ell = 2$  of the following more general conjecture.

**Conjecture 0.7.** *For partitions  $\pi_1, \dots, \pi_\ell$  we consider the expansion*

$$\kappa_\bullet(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_\ell}) = \sum_{\sigma} d_{\pi_1, \dots, \pi_\ell}^\sigma(\delta) \text{Ch}_\sigma.$$

*Then  $(-1)^{\ell-1} d_{\pi_1, \dots, \pi_\ell}^\sigma \in \mathbb{Q}[\delta]$  is a polynomial with non-negative integer coefficients.*

For example,

$$\begin{aligned} \kappa_\bullet(\text{Ch}_2, \text{Ch}_2, \text{Ch}_2) = \\ (8\delta^2 + 8) \text{Ch}_2 + 8\delta \text{Ch}_{1,1} + 64\delta \text{Ch}_3 + 64 \text{Ch}_{2,1} + 40 \text{Ch}_4. \end{aligned}$$

Clearly, Conjecture 0.7 gives support and some meaning to our vague Claim 0.5. Conjecture 0.7 also gives some support to the viewpoint that the coefficients  $d_{\pi_1, \dots, \pi_\ell}^\mu$  are a kind of *structure coefficients on steroids* which we shall adapt in the following.

#### 0.10. The main result: approximate factorization of Jack characters.

Our vague Claim 0.5 that the cumulants  $\kappa_\bullet$  have a *much nicer structure* than Jack characters themselves can be made concrete in yet another way which will be the core of the current paper. We start with an observation that the expression (0.8) for the cumulant is *less complex* and thus *more beautiful* than the corresponding expression (0.9) for the character. On a very silly level, comparison of complexity of two formulas is just a measurement of the amount of ink necessary to write each of them.

A more clever way is to define a *filtration* by declaring that the degrees of the variables are given by

$$\begin{aligned} \deg \mathcal{R}_n &= n & \text{for each } n \geq 2; \\ \deg \gamma &= 1. \end{aligned}$$

According to this approach, the complexity of an expression is measured by its degree according to this filtration.



In our example, the degree of the cumulant (0.8) is given by  $\deg \kappa_{\bullet}(\text{Ch}_2, \text{Ch}_2, \text{Ch}_2) = 5$  which is indeed much smaller than the degree of the corresponding moment (0.9), given by  $\deg \text{Ch}_{2,2,2} = 9$ .

The following theorem is the main result of the current paper.

**Theorem 0.8** (The main result: approximate factorization of Jack characters). *For any partitions  $\pi_1, \dots, \pi_\ell$  the corresponding cumulant is of much smaller degree than one would expect:*

$$\deg \kappa_{\bullet}(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_\ell}) \leq \deg \text{Ch}_{\pi_1 \dots \pi_\ell} - 2(\ell - 1).$$

We will prove this result in an equivalent but more fancy formulation as Theorem 1.9. We refer to this result as *approximate factorization* of characters because, informally speaking, it states that the character corresponding to the concatenation  $\pi_1 \cdots \pi_\ell$  of partitions is approximately equal to the product of the characters  $\text{Ch}_{\pi_i}$  corresponding to each of the partitions separately:

$$\text{Ch}_{\pi_1, \dots, \pi_\ell} \approx \text{Ch}_{\pi_1} \cdots \text{Ch}_{\pi_\ell}.$$

### 0.11. Motivations.

0.11.1. *Structure coefficients.* Until now we have recalled some old conjectures (Conjectures 0.4 and 0.6) and we have stated some new conjectures (Conjectures 0.1 and 0.7). From the perspective of an algebraic combinatorialist they provide a purely aesthetic motivation for investigation of the *structure coefficients on steroids*  $d_{\pi_1, \dots, \pi_\ell}^\mu$  and Kerov–Lassalle polynomials for  $\kappa_{\bullet}(\text{Ch}_a, \dots, \text{Ch}_z)$  with  $a, \dots, z \geq 1$ .

Our main result, Theorem 0.8 gives some partial answer to Conjecture 0.7, namely it gives some non-trivial upper bound on the degree of the polynomial  $d_{\pi_1, \dots, \pi_\ell}^\mu \in \mathbb{Q}[\delta]$  or, in other words, it shows that some concrete coefficients of this polynomial are indeed non-negative integers (to be more specific: zero). In the same spirit, Theorem 0.8 sheds also some light on Conjecture 0.6.

0.11.2. *Some special structure coefficients.* From the perspective of enumerative combinatorics, proving non-negativity and integrality of some numbers by showing that they are equal to zero, might be somewhat disappointing. On the bright side, our main Theorem 0.8 can be also used for some more positive enumerative results: in a forthcoming paper [Bur16] Burchardt gives an explicit combinatorial interpretation for the coefficients of high-degree monomials in the deformation parameter  $\delta$  for some special choices of the partitions. The simplest coefficient for which Burchardt's results are applicable is

$$[\delta^{k+l+1-m}] g_{(k), (l)}^{(m)}$$

for integers  $k, l, m \geq 1$ .

0.11.3. *Jack polynomials.* The motivations which we discussed above, fit all into the category “*no matter what Jack characters are and where they come from, they seem to give rise to interesting combinatorics*”. However, there are also some good self-standing reasons for studying Jack characters: they come from the fact that — as the name itself suggests — *Jack characters* are related to *Jack polynomials*.

Henry Jack [Jac71] introduced a family  $(J_\pi^{(\alpha)})$  (indexed by an integer partition  $\pi$ ) of symmetric functions which depend on an additional parameter  $\alpha$ . During the last forty years, many connections of these *Jack polynomials* with various fields of mathematics and physics were established: it turned out that they play a crucial role in understanding Ewens random permutations model [DH92], generalized  $\beta$ -ensembles and some statistical mechanics models [OO97], Selberg-type integrals [Kan93], certain random partition models [BO05], and some problems of the algebraic geometry [Oko03], among many others. Better understanding of Jack polynomials is also very desirable in the context of generalized  $\beta$ -ensembles and their discrete counterpart model [DF16]. Jack polynomials are a special case of the celebrated *Macdonald polynomials* which “*have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics*” [GR05].

We expand Jack polynomial in the basis of power-sum symmetric functions:

$$(0.10) \quad J_\lambda^{(\alpha)} = \sum_{\pi} \theta_\pi^{(\alpha)}(\lambda) p_\pi.$$

The above sum runs over partitions  $\pi$  such that  $|\pi| = |\lambda|$ . The coefficient  $\theta_\pi^{(\alpha)}(\lambda)$  is called *unnormalized Jack character*; with the right choice of the normalization it becomes *the normalized Jack character*  $\text{Ch}_\pi(\lambda)$  (the details of this relationship will be given in Definition 1.1). Thus Jack characters  $\theta_\pi^{(\alpha)}$  (respectively,  $\text{Ch}_\pi$ ) provide a kind of *dual* information about the Jack polynomials; a better understanding of the combinatorics of Jack characters may lead to a better understanding of Jack polynomials themselves.

0.11.4. *Connection coefficients for Jack symmetric functions.* A simplistic version of the structure coefficients which we study in the current paper can be traced back to the work of Goulden and Jackson [GJ96, Eqs. (1) and (5)] who considered the following expansion of the left-hand side in terms of

the power-sum symmetric functions:

$$\sum_{\theta \in \mathcal{P}} \frac{1}{\langle J_\theta, J_\theta \rangle_\alpha} J_\theta^{(\alpha)}(\mathbf{x}) J_\theta^{(\alpha)}(\mathbf{y}) J_\theta^{(\alpha)}(\mathbf{z}) t^{|\theta|} = \sum_{n \geq 1} t^n \sum_{\lambda, \mu, \nu \vdash n} \frac{c_{\mu, \nu}^\lambda}{\alpha^{\ell(\lambda)}} \frac{|\mathcal{C}_\lambda|}{n!} p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y}) p_\lambda(\mathbf{z}).$$

For the missing notation we refer to the original work of Goulden and Jackson. The coefficients  $c_{\mu, \nu}^\lambda$  in this expansion depend implicitly on the deformation parameter  $\alpha$ ; Goulden and Jackson argued that they can be viewed as a generalization of the *connection coefficients for the symmetric groups*.

It is worth pointing out that the coefficients  $(c_{\mu, \nu}^\lambda)$  are indexed by *three partitions of the same size* while the quantities  $(g_{\mu, \nu}^\lambda)$  considered in the current paper and in [DF16] are indexed by triples of *arbitrary partitions*. Dołęga and Féray [DF16, Section 4.2 and Appendix B.2] investigated the relationship between these two families of coefficients; in particular they found [DF16, Eq. (21)] an explicit formula which gives  $(c_{\mu, \nu}^\lambda)$  in terms of  $(g_{\mu, \nu}^\lambda)$ .

Thus a better understanding of the structure coefficients  $(g_{\mu, \nu}^\lambda)$  might shed some light on the open problems stated by Goulden and Jackson and, in particular, on *b*-conjecture [GJ96, Conjecture 3.5].

**0.11.5. Gaussian fluctuations for Jack-deformed random Young diagrams.** Our personal motivation for proving Theorem 0.8 comes from a forthcoming joint paper with Dołęga [DŚ16]. The latter paper has a purely probabilistic flavor: it concerns some *random Young diagrams related to Jack polynomials*. We show there that the fluctuations of these random Young diagrams around the limit shapes are asymptotically Gaussian. This result is a generalization of the results from our previous paper [Śni06a] to a much more general class of probability distributions related to Jack polynomials and Jack deformation. One of these probability distributions is, for example, Plancherel–Jack measure for which the Gaussianity of fluctuations was proved only very recently [DF16].

The main result of the current paper, Theorem 0.8, is the key technical tool which will be necessary in [DŚ16].

## 1. INTRODUCTION

**1.1. Partitions.** A *partition*  $\lambda = (\lambda_1, \dots, \lambda_l)$  is defined as a weakly decreasing finite sequence of positive integers. If  $\lambda_1 + \dots + \lambda_l = n$  we also say that  $\lambda$  is a *partition of  $n$* . We also define

$$|\lambda| := \lambda_1 + \dots + \lambda_l$$

and say that  $\ell(\lambda) := l$  is the *number of parts of  $\lambda$*  and that

$$m_i(\lambda) := |\{k : \lambda_k = i\}|$$

is the *multiplicity of  $i \geq 1$  in the partition  $\lambda$* .

For an integer  $n \geq 0$  we denote by  $\mathcal{P}_n$  the set of all partitions of the number  $n$ . We denote by

$$\mathcal{P} := \bigsqcup_{n \geq 0} \mathcal{P}_n$$

the set of all partitions.

When dealing with partitions we will use the shorthand notation

$$1^l := \underbrace{(1, \dots, 1)}_{l \text{ times}}.$$

For partitions  $\pi_1, \pi_2, \dots, \pi_k$  we define their *product*  $\pi_1 \cdots \pi_k$  as their concatenation.

The expression *Young diagram* is fully synonymous to the expression *partition*. However, for aesthetical reasons we will use each of them in a different context. Since *partitions* are used in order to enumerate conjugacy classes of the symmetric groups while *Young diagrams* in order to enumerate irreducible representations of the symmetric groups, similarly the Jack character  $\text{Ch}_\pi(\lambda)$  depends on the *partition*  $\pi$  and on the *Young diagram*  $\lambda$ .

The empty partition as well as the empty Young diagram will be denoted by the same symbol  $\emptyset$ .

**1.2. Set-partitions.** We say that  $\nu = \{\nu_1, \dots, \nu_l\}$  is a *set-partition* of some set  $X$  if  $\nu_1, \dots, \nu_l$  are disjoint, non-empty sets such that

$$\bigsqcup_{1 \leq i \leq l} \nu_i = X.$$

We refer to the sets  $\nu_1, \dots, \nu_l$  as *blocks* of the set-partition  $\nu$ .

**1.3. Normalized Jack characters, the first definition.** In this section we shall present the definition of Jack characters  $\text{Ch}_\pi$  which appeared historically as the first one and which is based on the notion of *Jack polynomials*. The results of this paper will not refer to this definition; for this reason the Readers faint at heart, who do not appreciate Jack polynomials too much may fast forward to Section 1.4 where we start preparation for an alternative, equivalent, self-contained definition of  $\text{Ch}_\pi$ .

The usual way of viewing the characters of the symmetric groups is to fix the representation  $\lambda$  and to consider the character as a function of the conjugacy class  $\pi$ . However, there is also another very successful viewpoint due to Kerov and Olshanski [KO94], called *dual approach*, which suggests

to do roughly the opposite. Lassalle [Las08, Las09a] adapted this idea to the framework of Jack characters.

In order for this dual approach to be successful (both with respect to the usual characters of the symmetric groups and for the Jack characters) one has to choose the most convenient normalization constants. In the current paper we will use the normalization introduced by Dołęga and Féray [DF16] which offers some advantages over the original normalization of Lassalle. Thus, with the right choice of the multiplicative constant, the unnormalized Jack character  $\theta_\lambda^{(\alpha)}(\pi)$  from (0.10) becomes the *normalized Jack character*  $\text{Ch}_\pi^{(\alpha)}(\lambda)$ , defined as follows.

**Definition 1.1.** Let  $\alpha > 0$  be given and let  $\pi$  be a partition. The *normalized Jack character*  $\text{Ch}_\pi^{(\alpha)}(\lambda)$  is given by:

$$(1.1) \quad \text{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} \alpha^{-\frac{|\pi|-\ell(\pi)}{2}} \binom{|\lambda|+|\pi|+m_1(\pi)}{m_1(\pi)} z_\pi \theta_{\pi, 1^{|\lambda|-|\pi|}}^{(\alpha)}(\lambda) & \text{if } |\lambda| \geq |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|, \end{cases}$$

where

$$z_\pi = \prod_i i^{m_i(\pi)} m_i(\pi)!$$

is the standard numerical factor. The choice of an empty partition  $\pi = \emptyset$  is acceptable; in this case  $\text{Ch}_\emptyset^{(\alpha)}(\lambda) = 1$ .

The above definition and, in particular, the choice of the normalization factors may at the first sight appear repulsive even for the Readers who appreciate Jack polynomials. Later on, in Definition 1.5 we shall see that thanks to this choice of the normalization Jack characters allow a convenient alternative description.

**1.4. The deformation parameters.** In order to avoid dealing with the square root of the variable  $\alpha$  we introduce an indeterminate  $A$  such that

$$A^2 = \alpha.$$

*Jack characters* are usually defined in terms of the deformation parameter  $\alpha$ . After the substitution  $\alpha := A^2$  each Jack character becomes a function of  $A$ ; in order to keep the notation light we will make this dependence implicit and we will simply write  $\text{Ch}_\pi(\lambda)$ .

The algebra of Laurent polynomials in the indeterminate  $A$  will be denoted by  $\mathbb{Q}[A, A^{-1}]$ .

**Definition 1.2.** For an integer  $d$  we will say that a *Laurent polynomial*

$$f = \sum_{k \in \mathbb{Z}} f_k A^k \in \mathbb{Q}[A, A^{-1}]$$

is of degree at most  $d$  if  $f_k = 0$  holds for each integer  $k > d$ .

A special role will be played by the quantity

$$(1.2) \quad \gamma := -A + \frac{1}{A} \in \mathbb{Q}[A, A^{-1}].$$

**1.5.  $\alpha$ -content.** For drawing Young diagrams we use the French convention and the usual Cartesian coordinate system; in particular, the box  $(x, y) \in \mathbb{N}^2$  is the one in the intersection of the column with the index  $x$  and the row with the index  $y$ . We index the rows and the columns by the elements of the set

$$\mathbb{N} = \{1, 2, \dots\}$$

of positive integers; in particular the first row as well as the first column correspond to the number 1.

*Definition 1.3.* For a box  $\square = (x, y)$  of a Young diagram we define its  $\alpha$ -content by

$$\alpha\text{-content}(\square) = \alpha\text{-content}(x, y) := Ax - \frac{1}{A}y \in \mathbb{Q}[A, A^{-1}].$$

**1.6. The algebra  $\mathcal{P}$  of  $\alpha$ -polynomial functions, the first definition.** In our recent paper [Śni15, Definition 1.5] we have defined a filtered algebra  $\mathcal{P}$ , which is called *the algebra of  $\alpha$ -polynomial functions*. This algebra consists of certain functions on the set  $\mathbb{Y}$  of Young diagrams with values in the ring  $\mathbb{Q}[A, A^{-1}]$  of Laurent polynomials. The following definition specifies  $\mathcal{P}$  as a set and defines the filtration on it.

*Definition 1.4.* For an integer  $d \geq 0$  we say that

$$F: \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$$

is an  $\alpha$ -polynomial function of degree at most  $d$  if there exists a sequence  $p_0, p_1, \dots$  of polynomials which satisfies the following properties:

- for each  $k \geq 0$  we have that  $p_k \in \mathbb{Q}[\gamma, c_1, \dots, c_k]$  is a polynomial in the variables  $\gamma, c_1, \dots, c_k$  of degree at most  $d - 2k$ ;
- for each Young diagram  $\lambda$ ,

$$F(\lambda) = \sum_{k \geq 0} \sum_{\square_1, \dots, \square_k \in \lambda} p_k(\gamma, c_1, \dots, c_k) \in \mathbb{Q}[A, A^{-1}],$$

where the second sum runs over all tuples of boxes of the Young diagram; furthermore

$$c_1 := \alpha\text{-content}(\square_1), \quad \dots, \quad c_k := \alpha\text{-content}(\square_k)$$

are the corresponding  $\alpha$ -contents, and the substitution (1.2) is used.

The multiplication in  $\mathcal{P}$  was understood in [Śni15] as the pointwise product of the functions.

**1.7. Jack characters, the second definition.** One of the main results of our recent work [Śni15, Theorem 1.7] is the equivalence between Definition 1.1 and the following abstract definition of Jack characters which does not refer to the notion of Jack polynomials.

*Definition 1.5.* Jack character  $\text{Ch}_\pi$  is the unique  $\alpha$ -polynomial function  $F$  which fulfills the following properties:

- (K1)  $F \in \mathcal{P}$  is an  $\alpha$ -polynomial function of degree at most  $|\pi| + \ell(\pi)$ ;
- (K2) for each  $m \geq 1$  the function in  $m$  variables

$$\mathbb{Y} \ni (\lambda_1, \dots, \lambda_m) \mapsto F(\lambda_1, \dots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]$$

is a polynomial of degree  $|\pi|$  and its homogeneous top-degree part is equal to

$$A^{|\pi| - \ell(\pi)} p_\pi(\lambda_1, \dots, \lambda_m),$$

where  $p_\pi$  is the *power-sum symmetric polynomial*;

- (K3) for each Young diagram  $\lambda \in \mathbb{Y}$  such that  $|\lambda| < |\pi|$  we have

$$F(\lambda) = 0;$$

- (K4) for each Young diagram  $\lambda \in \mathbb{Y}$  the evaluation  $F(\lambda) \in \mathbb{Q}[A, A^{-1}]$  is a Laurent polynomial of degree at most  $|\pi| - \ell(\pi)$ .

Roughly speaking, conditions (K1), (K2) and (K4) specify the asymptotic behavior of the character  $\text{Ch}_\pi(\lambda)$  in three kinds of limits:

- when the shape of the Young diagram  $\lambda$  tends to infinity together with the variable  $\gamma$ ,
- when the rows  $\lambda_1, \lambda_2, \dots$  of the Young diagram tend to infinity,
- when the deformation parameter  $A$  tends to infinity;

on the other hand condition (K3) is of quite different flavor because it concerns the values of the Jack character on small Young diagrams.

*Example 1.6.* By easily checking that the conditions from Definition 1.5 are fulfilled, one can verify the following equalities:

$$\begin{aligned}
\text{Ch}_\emptyset(\lambda) &= 1, \\
\text{Ch}_1(\lambda) &= \sum_{\square_1 \in \lambda} 1, \\
\text{Ch}_2(\lambda) &= \sum_{\square_1 \in \lambda} 2(c_1 + \gamma), \\
\text{Ch}_3(\lambda) &= \sum_{\square_1 \in \lambda} \left( 3(c_1 + \gamma)(c_1 + 2\gamma) + \frac{3}{2} \right) + \sum_{\square_1, \square_2 \in \lambda} \left( -\frac{3}{2} \right), \\
\text{Ch}_{1,1}(\lambda) &= \sum_{\square_1 \in \lambda} (-1) + \sum_{\square_1, \square_2 \in \lambda} 1,
\end{aligned}$$

where  $\gamma$  is given by (1.2).

### 1.8. The algebra $\mathcal{P}$ of $\alpha$ -polynomial functions, the second definition.

**Proposition 1.7** ([Śni15, Proposition 2.17]). *The family of functions*

$$(1.3) \quad \gamma^p \text{Ch}_\pi : \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$$

*over integers  $p \geq 0$  and partitions  $\pi \in \mathcal{P}$  is linearly independent over  $\mathbb{Q}$ .*

*The linear space of the elements of  $\mathcal{P}$  of degree at most  $d$  is spanned (over  $\mathbb{Q}$ ) by the elements (1.3) such that*

$$p + |\pi| + \ell(\pi) \leq d.$$

In other words, if we take Jack characters  $\text{Ch}_\pi$  as a primitive notion (for example, by using Definition 1.1) one can *define* the linear structure of  $\mathcal{P}$  as the free  $\mathbb{Q}[\gamma]$ -module with the linear basis given by Jack characters  $\text{Ch}_\pi$ . With this approach, the degrees of the generators are given by

$$(1.4) \quad \begin{cases} \deg \text{Ch}_\pi = |\pi| + \ell(\pi), \\ \deg \gamma = 1. \end{cases}$$

### 1.9. Algebra $\mathcal{P}_\bullet$ of $\alpha$ -polynomial functions with the disjoint product $\bullet$ .

The set  $\mathcal{P}$  of  $\alpha$ -polynomial functions can be equipped with another multiplication  $\bullet$ , called *disjoint product*, which is defined on the linear base of Jack characters by *concatenation* (see the end of Section 1.1) of the corresponding partitions

$$(\gamma^p \text{Ch}_\pi) \bullet (\gamma^q \text{Ch}_\sigma) := \gamma^{p+q} \text{Ch}_{\pi\sigma}.$$

It is easy to check that this product is commutative and associative; the set of  $\alpha$ -polynomial functions equipped with this multiplication becomes an algebra which will be denoted by  $\mathcal{P}_\bullet$ . It is easy to check that the usual



filtration (1.4) works fine also with this product; in this way  $\mathcal{P}_\bullet$  becomes a filtered algebra.

**1.10. Conditional cumulants.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be commutative unital algebras and let  $\mathbb{E}: \mathcal{A} \rightarrow \mathcal{B}$  be a unital linear map. We will say that  $\mathbb{E}$  is a *conditional expectation value*; in the literature one usually imposes some additional constraints on the structure of  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathbb{E}$ , but for the purposes of the current paper such additional assumptions will not be necessary.

For any tuple  $x_1, \dots, x_n \in \mathcal{A}$  we define their *conditional cumulant* as

$$\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, \dots, x_n) = [t_1 \cdots t_n] \log \mathbb{E} e^{t_1 x_1 + \cdots + t_n x_n} = \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E} e^{t_1 x_1 + \cdots + t_n x_n} \right|_{t_1 = \cdots = t_n = 0} \in \mathcal{B}$$

where the operations on the right-hand side should be understood in the sense of formal power series in variables  $t_1, \dots, t_n$ .

**1.11. Approximate factorization property.** The following definition is key for the current paper.

*Definition 1.8.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be filtered unital algebras and let  $\mathbb{E}: \mathcal{A} \rightarrow \mathcal{B}$  be a conditional expectation; we denote by  $\kappa_{\mathcal{A}}^{\mathcal{B}}$  the corresponding cumulants. We say that  $\mathbb{E}$  *has approximate factorization property* if for all choices of  $x_1, \dots, x_l \in \mathcal{A}$  we have that

$$\deg \kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, \dots, x_l) \leq \deg x_1 + \cdots + \deg x_l - 2(l-1).$$

**1.12. The main result: conditional cumulants between the disjoint and the usual product.** We consider the filtered unital algebras  $\mathcal{P}_\bullet$  and  $\mathcal{P}$ , and as a conditional expectation between them we take the identity map:

$$(1.5) \quad \mathcal{P}_\bullet \xrightarrow{\text{id}} \mathcal{P}.$$

This structure may appear to be misleadingly simple. Note, however, that even though  $\mathcal{P}_\bullet$  and  $\mathcal{P}$  are equal as *vector spaces*, they are furnished with different multiplication structures: the disjoint product  $\bullet$  and the usual pointwise product of functions, respectively. For this reason this identity map is far from being trivial.

We denote by  $\kappa_\bullet$  the conditional cumulants related to the conditional expectation (1.5). Informally speaking, this cumulant quantifies how *big* is the discrepancy between the two multiplications on the set of  $\alpha$ -polynomial functions: the disjoint product  $\bullet$  and the usual pointwise multiplication of functions.

Based on Proposition 1.7, the main result of the current paper, Theorem 0.8, can be equivalently reformulated in the following more abstract way.

**Theorem 1.9** (Equivalent formulation of the main result). *The identity map*

$$\mathcal{P}_\bullet \xrightarrow{\text{id}} \mathcal{P}.$$

*has approximate factorization property.*

*In other words, for any partitions  $\pi_1, \dots, \pi_l \in \mathcal{P}$  the corresponding conditional cumulant*

$$\kappa_\bullet(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}) \in \mathcal{P}$$

*is of degree at most*

$$|\pi_1| + \dots + |\pi_l| + \ell(\pi_1) + \dots + \ell(\pi_l) - 2(l-1).$$

The proof will be presented in Section 2.11.

### 1.13. History of the result.

#### 1.13.1. Approximate factorization for the characters of the symmetric groups.

The normalized Jack characters  $\text{Ch}_\pi^{(\alpha)}$  are a generalization of the usual normalized characters of the symmetric groups  $\text{Ch}_\pi^{(1)}$  [Las09a, DF16]; one of the manifestations of this phenomenon was discussed in Fact 0.2. This classical context when the deformation parameter is specialized to  $\alpha = 1$  carries more algebraic, representation-theoretic and combinatorial structures than the general case of Jack characters and Jack polynomials; for this reason much more is known in this case. We shall review this special case  $\alpha = 1$  from the perspective of *approximate factorization of Jack characters* (Theorem 1.9).

First of all, the algebra  $\mathcal{P}$  for  $\alpha = 1$  still makes sense but it becomes an algebra of functions on the set  $\mathbb{Y}$  of Young diagrams with values in the rational numbers  $\mathbb{Q}$ . We denote this algebra by  $\mathcal{P}^{(1)}$ , respectively by  $\mathcal{P}_\bullet^{(1)}$  in the case when as the multiplication we take the disjoint product  $\bullet$ . There is no need to speak about the indeterminate  $\gamma$  (which becomes now a constant, equal to zero). Also our definition of the filtration on  $\mathcal{P}$  is remains valid for  $\alpha = 1$ ; thus  $\mathcal{P}^{(1)}$  and  $\mathcal{P}_\bullet^{(1)}$  are filtered algebras. As a consequence, the statement of Theorem 1.9 is well-defined in this case and takes the following concrete form.

**Theorem 1.10.** *The identity map*

$$(1.6) \quad \mathcal{P}_\bullet^{(1)} \xrightarrow{\text{id}} \mathcal{P}^{(1)}.$$

*has approximate factorization property.*

*In other words, for any partitions  $\pi_1, \dots, \pi_l \in \mathcal{P}$  the corresponding conditional cumulant*

$$\kappa_\bullet\left(\text{Ch}_{\pi_1}^{(1)}, \dots, \text{Ch}_{\pi_l}^{(1)}\right) \in \mathcal{P}^{(1)}$$

is a linear combination (with rational coefficients) of normalized characters of the symmetric groups  $\text{Ch}_\pi^{(1)}$  over partitions  $\pi$  such that

$$|\pi| + \ell(\pi) \leq |\pi_1| + \cdots + |\pi_l| + 2 - l.$$

This result is much simpler than our original goal, Theorem 1.9 and several of its proofs are available; we shall review them in the following.

1.13.2. *Approximate factorization via multiplication of partial permutations.* Consider the conditional expectation defined as the inverse of the identity map from (1.6):

$$(1.7) \quad \mathcal{P}_\bullet^{(1)} \xleftarrow{\text{id}} \mathcal{P}^{(1)}.$$

We shall denote by  $\kappa^\bullet$  the corresponding cumulants.

*Remark 1.11.* There are two different cumulants (namely  $\kappa_\bullet$  and  $\kappa^\bullet$ ) which measure the discrepancy between the two products which are available in  $\mathcal{P}^{(1)}$ . This might be quite confusing and, indeed, there is some confusion in the literature around this issue. As far as we know, the cumulant  $\kappa^\bullet$  is considered only in the papers [Śni06a, Śni06b] while the cumulant  $\kappa_\bullet$  is considered in the remaining articles in the field, including [RŚ08, Fér09, DFŚ10, FŚ11]. Regretfully, the papers [DFŚ10, Fér09] wrongfully claim that [Śni06a] concern the cumulants  $\kappa_\bullet$ .

The difference between  $\kappa_\bullet$  and  $\kappa^\bullet$  is that the roles played by the disjoint product and the pointwise product are interchanged. For example,

$$\kappa_\bullet(\text{Ch}_{\pi_1}, \text{Ch}_{\pi_2}) = \text{Ch}_{\pi_1} \bullet \text{Ch}_{\pi_2} - \text{Ch}_{\pi_1} \text{Ch}_{\pi_2} = \text{Ch}_{\pi_1, \pi_2} - \text{Ch}_{\pi_1} \text{Ch}_{\pi_2}$$

while

$$\kappa^\bullet(\text{Ch}_{\pi_1}, \text{Ch}_{\pi_2}) = \text{Ch}_{\pi_1} \text{Ch}_{\pi_2} - \text{Ch}_{\pi_1} \bullet \text{Ch}_{\pi_2} = \text{Ch}_{\pi_1} \text{Ch}_{\pi_2} - \text{Ch}_{\pi_1, \pi_2}.$$

For a larger number of arguments the difference between  $\kappa_\bullet$  and  $\kappa^\bullet$  becomes more complicated than just a change of the sign.

We previously proved the the following result.

**Theorem 1.12** ([Śni06a, Theorem 15]). *The map (1.7) has approximate factorization property.*

Formally speaking, Theorem 1.10 in its exact formulation does not appear in [Śni06a], but it is not very difficult to use Brillinger's formula (Lemma 2.11 below) in order to show that Theorem 1.10 and Theorem 1.12 are equivalent. We leave the details to the Reader.

The proof of Theorem 1.12 from [Śni06a] was based on the ideas which we discussed in Sections 0.2 and 0.3. More specifically, for each integer

$n \geq 1$  the algebra of functions on the set of Young diagrams with  $n$  boxes is isomorphic (via noncommutative Fourier transform) to the center of the symmetric group algebra  $\mathbb{C}[\mathfrak{S}(n)]$  with the product given by convolution. The image of the character  $\text{Ch}_\pi^{(1)}$  under this isomorphism is equal, up to some simple multiplicative constant, to the indicator function of the conjugacy class in  $\mathfrak{S}(n)$  of the permutations with the cycle decomposition given by  $\pi$ . This means that the problem of understanding the pointwise multiplication of functions on  $\mathbb{Y}$  can be translated into the problem of understanding convolution of conjugacy classes in the symmetric group algebra.

This observation can be further improved if, instead of the usual permutations, one uses *partial permutations* of Ivanov and Kerov [IK99]. In this way the above isomorphism becomes a map between the algebra  $\mathcal{P}^{(1)}$  and a certain semigroup algebra of partial permutations. The advantage of this viewpoint comes from the fact that the disjoint product in  $\mathcal{P}_\bullet^{(1)}$  corresponds under this isomorphism to the *disjoint product of partial permutations*. This means that our original problem of understanding the relationship between the pointwise and the disjoint product in  $\mathcal{P}$  is equivalent to finding the analogous relationship between the convolution product and the disjoint product in the algebra of partial permutations and for the latter problem one can find some explicit formulas.

One can ask whether this strategy of proof could be used to prove our goal, Theorem 1.9. In order to do this we would have to find some *concrete, convenient algebra* (which should be some generalization and deformation of the group algebra of the symmetric group, respectively, the algebra of the semigroup of partial permutations), the representation theory of which would be given by Jack characters. Existence of such an algebra is the content of Conjecture 0.3.

Regretfully, not much is known about the combinatorics of the structure coefficients for Jack characters [DF16, Bur16]; the structure of the hypothetical deformation of the group algebra of the symmetric group remains even more elusive. Thus this path of proving Theorem 1.9 is currently not available.

**1.13.3. Approximate factorization via explicit formulas for the characters.** Another way of proving Theorem 1.10 is to start with some formula for the characters  $\text{Ch}_\pi^{(1)}$  of the symmetric groups. A good choice is a formula which was conjectured by Stanley [Sta06], proved by Féray [Fér09] and reformulated in a convenient way in our joint paper with Féray [FŚ11, Theorem 2]. The second step is to use the above formula in order to get a closed formula for the cumulants  $\kappa_\bullet$ , see [Fér09, Section 1.6] and [DFŚ10, Theorem 4.7].

The third and the final step is to use the latter formula for  $\kappa_\bullet$  in order to find a suitable upper bound for the degree of these cumulants.

Regretfully, in the case of general Jack characters  $\text{Ch}_\pi^{(\alpha)}$  the problem of finding a convenient closed formula still remains elusive [DFS14, Śni15] and this path of proving Theorem 1.9 is currently not available.

**1.14. How to prove the main result? New product  $\otimes$ .** We start with a heuristic overview of the general strategy for proving Theorem 1.9.

**1.14.1. The vanishing property.** Condition (K3) in the abstract definition of Jack character  $\text{Ch}_\pi$  (Definition 1.5) states that

$$(1.8) \quad \text{Ch}_\pi(\lambda) = 0 \quad \text{if } |\lambda| < |\pi|.$$

Clearly, this *vanishing property* is a trivial consequence of the way normalized Jack characters were defined via Jack polynomials (Definition 1.1); the interesting part is that (1.8) can be seen as a system of a sufficient number of linear equations which is necessary in order to characterize the Jack character uniquely.

This observation gives rise to the following natural question: *would it be possible to mimic these ideas and to find an abstract characterization of the cumulants  $\kappa_\bullet$  which would be analogous to Definition 1.5?* Regretfully, already in the simplest non-trivial example

$$(1.9) \quad \kappa_\bullet(\text{Ch}_a, \text{Ch}_b) = \text{Ch}_{a,b} - \text{Ch}_a \text{Ch}_b,$$

where  $a, b \geq 1$  are some integers we encounter a serious difficulty: the vanishing condition (1.8) for Jack characters implies that the analogous condition for the cumulant

$$(1.10) \quad (\kappa_\bullet(\text{Ch}_a, \text{Ch}_b))(\lambda) = 0$$

holds true only for relatively small Young diagrams  $\lambda$  such that  $|\lambda| < \max(a, b)$ . In other words, (1.10) gives rise to too few linear equations in order to characterize  $\kappa_\bullet(\text{Ch}_a, \text{Ch}_b)$  uniquely. For our purposes it would be more preferable to have (1.10) (or some its analogue) for all Young diagrams  $\lambda$  such that  $|\lambda| < a + b$ .

**1.14.2. New product  $\otimes$ .** The source of our problems is the pointwise product of functions which enters the second summand on the right-hand side of (1.9). This motivates investigation of some new product  $\otimes$  of functions on  $\mathbb{Y}$  which would have the following desirable property.

**Property 1.13.** Assume that  $a, b \geq 0$  are integers and  $F, G: \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$  are functions on the set  $\mathbb{Y}$  of Young diagrams such that

$$(1.11) \quad \begin{aligned} F(\lambda) = 0 & \quad \text{holds true for all } \lambda \in \mathbb{Y} \text{ such that } |\lambda| < a, \\ G(\lambda) = 0 & \quad \text{holds true for all } \lambda \in \mathbb{Y} \text{ such that } |\lambda| < b. \end{aligned}$$

Then

$$(F \otimes G)(\lambda) = 0 \quad \text{holds true for all } \lambda \in \mathbb{Y} \text{ such that } |\lambda| < a + b.$$

A convenient example of a product with this property, *the separate product*, will be presented in Section 2.4. This product gives rise to new cumulants  $\kappa_{\bullet}^{\otimes}$ , for example:

$$\kappa_{\bullet}^{\otimes}(\text{Ch}_a, \text{Ch}_b) = \text{Ch}_{a,b} - \text{Ch}_a \otimes \text{Ch}_b.$$

As a consequence of Property 1.13, the cumulants  $\kappa_{\bullet}^{\otimes}$  have an advantage that an analogue of the vanishing property

$$(1.12) \quad (\kappa_{\bullet}^{\otimes}(\text{Ch}_a, \text{Ch}_b))(\lambda) = 0$$

holds true for all Young diagrams such that  $|\lambda| < a + b$ ; in other words (1.12) gives a sufficient number of linear equations in order to understand the cumulants  $\kappa_{\bullet}^{\otimes}$  well.

**1.14.3. Three filtered algebras.** Regretfully, the cumulants  $\kappa_{\bullet}^{\otimes}$  are not the ones which we want to understand in Theorem 1.9. For this reason we will need to relate the properties of the cumulants  $\kappa_{\bullet}^{\otimes}$  to the cumulants  $\kappa_{\bullet}$  and to show that they are not far one from the other.

To summarize; we consider three products of functions on  $\mathbb{Y}$ , namely: the disjoint product  $\bullet$ , the pointwise product, and the separate product  $\otimes$ . These three products correspond to the three algebras in the following commutative diagram:

$$(1.13) \quad \begin{array}{ccc} \mathcal{P}_{\bullet} & \xrightarrow{\text{id}} & \mathcal{P} \\ & \searrow \text{id} & \downarrow \text{id} \\ & & \mathcal{R}_{\otimes} \end{array}$$

Each of the three arrows in this diagram can be viewed as a conditional expectation between commutative unital algebras, thus it gives rise to some conditional cumulants. Our original goal (proving Theorem 1.9) corresponds to understanding the cumulants  $\kappa_{\bullet}$  which correspond to the horizontal arrow. However, the discussion from the above Section 1.14.2 shows that the cumulants  $\kappa_{\bullet}^{\otimes}$  which correspond to the diagonal arrow have some

advantages. Our general strategy will be make advantage of the commutative diagram (1.13) and of the nice properties of the cumulants which correspond to the diagonal arrow and to the vertical one.

## 2. PROOF OF THEOREM 1.9

**2.1. Towards the proof: Lemma 2.3.** The proof of Theorem 1.9 is centered around Lemma 2.3 which was stated and proved in our recent paper [Śni15, Theorem 7.2] for quite different purposes, but it turns out to be also well-suited for the problems studied in the current paper. Roughly speaking, Lemma 2.3 provides a criterion for verifying that a given  $\alpha$ -polynomial function  $F \in \mathcal{P}$  which *a priori* is of some high degree, in fact turns out to be of some smaller degree (which is, obviously, the context of Theorem 1.9).

We shall start in Section 2.2 by gathering the definitions which are necessary to *state* Lemma 2.3. In Section 2.3 we state the key Lemma 2.3. In Sections 2.4–2.10 we will verify that the assumptions of Lemma 2.3 are indeed fulfilled in our context of the proof of Theorem 1.9. Finally, in Section 2.11 we apply Lemma 2.3 and in this way we prove Theorem 1.9.

### 2.2. Notations.

**2.2.1. Extension of the domain of functions on  $\mathbb{Y}$ .** Let  $F$  be a function on the set of Young diagrams. Such a function can be viewed as a function  $F(\lambda_1, \dots, \lambda_\ell)$  defined for all non-negative integers  $\lambda_1 \geq \dots \geq \lambda_\ell$ . We will extend its domain, as follows.

*Definition 2.1.* If  $(\xi_1, \dots, \xi_\ell)$  is an arbitrary sequence of non-negative integers, we denote

$$F^{\text{sym}}(\xi_1, \dots, \xi_\ell) := F(\lambda_1, \dots, \lambda_\ell),$$

where  $(\lambda_1, \dots, \lambda_\ell) \in \mathbb{Y}$  is the sequence  $(\xi_1, \dots, \xi_\ell)$  sorted in the reverse order  $\lambda_1 \geq \dots \geq \lambda_\ell$ . In this way  $F^{\text{sym}}(\xi_1, \dots, \xi_\ell)$  is a symmetric function of its arguments.

### 2.2.2. The difference operator.

*Definition 2.2.* If  $F = F(\lambda_1, \dots, \lambda_\ell)$  is a function of  $\ell$  arguments and  $1 \leq j \leq \ell$ , we define a new function  $\Delta_{\lambda_j} F$  by

$$\begin{aligned} (\Delta_{\lambda_j} F)(\lambda_1, \dots, \lambda_\ell) := \\ F(\lambda_1, \dots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \dots, \lambda_\ell) - F(\lambda_1, \dots, \lambda_\ell). \end{aligned}$$

We call  $\Delta_{\lambda_j}$  a *difference operator*.

**2.3. The key lemma.** The assumptions of the following lemma may appear complicated. Condition (Z1), (Z2) and (Z2) have been modeled after the analogous conditions from the abstract characterization of Jack characters (Definition 1.5). Only condition (Z3) is more mysterious: it turns out to be a version of (Z3) crafted in such a way that condition (Z3) holds true for any  $\alpha$ -polynomial function  $F \in \mathcal{P}$  which is of order at most  $n + r - 1$  (see Lemma 2.13).

**Lemma 2.3** ([Śni15, Theorem 7.2]). *Let integers  $n \geq 0$  and  $r \geq 1$  be given and let  $F: \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$  be a function on the set of Young diagrams. Assume that:*

(Z1)  $F \in \mathcal{P}$  is of degree at most  $n + r$ ;

(Z2) for each  $m \geq 1$  the function in  $m$  variables

$$\mathbb{Y} \ni (\lambda_1, \dots, \lambda_m) \mapsto F(\lambda_1, \dots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]$$

is a polynomial of degree at most  $n - 1$ ;

(Z3) the equality

$$(2.1) \quad [A^{n+r-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^{\text{sym}}(\lambda_1, \dots, \lambda_k) = 0$$

holds true for the following values of  $k$  and  $\lambda$ :

- $k = r$  and  $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Y}$  with at most  $r$  rows is such that  $|\lambda| \leq n + r - 2k - 1$ ;
- $k > r$  and  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Y}$  with at most  $k$  rows is such that  $|\lambda| \leq n + r - 2k$ ;

(Z4) for each  $\lambda \in \mathbb{Y}$ , the Laurent polynomial  $F(\lambda) \in \mathbb{Q}[A, A^{-1}]$  is of degree at most  $n + 1 - r$ .

Then  $F \in \mathcal{P}$  is of degree at most  $n + r - 1$ .

Alternative version: the result remains valid for all integers  $n \geq 0$  and  $r \geq 0$  if the assumption (Z2) is removed and the condition (Z3) is replaced by the following one:

(Z3a) the equality (2.1) holds true for all  $k \geq r$  and  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Y}$  with at most  $k$  rows such that  $|\lambda| \leq n + r - 2k$ .

As we shall see, only assumption (Z3) will turn out to be troublesome in our context of the proof of Theorem 1.9. Sections 2.4–2.10 will be devoted entirely to proving that this assumption indeed holds true.

**2.4. The strategy: row functions.** Our strategy in showing that the assumption (Z3) holds true will be based on the ideas which we already discussed in Section 1.14. Namely, we will introduce a convenient filtered vector space which consists of some functions on the set  $\mathbb{Y}$  of Young diagrams, with values in Laurent polynomials  $\mathbb{Q}[A, A^{-1}]$ . This linear space of



row functions can be equipped with the following two distinct multiplication structures.

Firstly, as a product we may take the usual pointwise multiplication of functions. In this way we obtain the filtered algebra  $\mathcal{R}$  which was introduced and studied in our recent paper [Śni15, Section 6]. Heuristically, the algebra  $\mathcal{R}$  consists of the functions which behave with respect to condition (Z3) in an analogous way as  $\alpha$ -polynomial functions. In particular, we shall prove that  $\mathcal{P} \subseteq \mathcal{R}$ , furthermore the multiplicative structure and the filtration structure of both algebras are compatible with each other. In other words, the passage from  $\mathcal{P}$  to  $\mathcal{R}$  causes no difficulties.

Secondly, as a product on the linear space of row functions we may take a new *separate product*  $\otimes$  which was crafted in such a way that Property 1.13 holds true. This gives rise to a filtered algebra  $\mathcal{R}_\otimes$  which is the bottom node in the diagram (1.13).

Thus, we have two multiplicative structures on the vector space of row functions. This naturally calls for investigation of the cumulants related to the conditional expectation

$$(2.2) \quad \begin{array}{c} \mathcal{R} \\ \downarrow \text{id} \\ \mathcal{R}_\otimes \end{array}$$

which would measure the discrepancy between these two multiplicative structures. Because of the nice properties of the inclusion  $\mathcal{P} \subseteq \mathcal{R}$  investigation of the conditional expectation (2.2) is almost the same as investigation of the vertical arrow in (1.13) which is our intermediate goal for proving that condition (Z3) holds true.

We shall present the details in the following.

## 2.5. Row functions.

*Definition 2.4.* Let a sequence (indexed by  $r \geq 0$ ) of symmetric functions  $f_r : \mathbb{N}_0^r \rightarrow \mathbb{Q}[A, A^{-1}]$  be given, where

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

We assume that:

- if  $0 \in \{x_1, \dots, x_r\}$  then  $f_r(x_1, \dots, x_r) = 0$ ,
- $f_r = 0$  except for finitely many values of  $r$ ,

- there exists some integer  $d \geq 0$  with the property that for all  $r \geq 0$  and all  $x_1, \dots, x_r \in \mathbb{N}_0$ , the evaluation  $f_r(x_1, \dots, x_r) \in \mathbb{Q}[A, A^{-1}]$  is a Laurent polynomial of degree at most  $d - 2r$ .

We define a function  $F: \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}]$  given by

$$(2.3) \quad F(\lambda) := \sum_{r \geq 0} \sum_{i_1 < \dots < i_r} f_r(\lambda_{i_1}, \dots, \lambda_{i_r}).$$

We will say that  $F$  is a *row function of degree at most  $d$*  and that  $(f_r)$  is the *convolution kernel* of  $F$ .

As we already mentioned, we define  $\mathcal{R}$  as the set of row functions equipped with the pointwise product of functions; one can show [Śni15, Lemma 6.4] that with this product  $\mathcal{R}$  becomes a filtered algebra.

We define  $\mathcal{R}_{\otimes}$  as the set of row functions equipped with *the separate product*  $\otimes$ . It is defined on the linear basis by declaring

$$\begin{aligned} \sum_{i_1 < \dots < i_l} f(\lambda_{i_1}, \dots, \lambda_{i_l}) \otimes \sum_{j_1 < \dots < j_m} g(\lambda_{j_1}, \dots, \lambda_{j_m}) := \\ \sum_{k_1 < \dots < k_{l+m}} \sum_{\substack{i_1 < \dots < i_l \\ j_1 < \dots < j_m \\ \{k_1, \dots, k_{l+m}\} = \{i_1, \dots, i_l\} \sqcup \{j_1, \dots, j_m\}}} f(\lambda_{i_1}, \dots, \lambda_{i_l}) g(\lambda_{j_1}, \dots, \lambda_{j_m}); \\ \underbrace{\hspace{15em}}_{h_{l+m}(\lambda_{k_1}, \dots, \lambda_{k_{l+m}})} \end{aligned}$$

we extend this definition by bilinearity to general elements of  $\mathcal{R}_{\otimes}$ . In other words, the separate product is defined by a kind of a tensor product of the corresponding convolution kernels. This product is well-defined since the convolution kernel is uniquely determined by the row function (see [Śni15, Remark 6.2]).

It is a very simple exercise to check that the algebra  $\mathcal{R}_{\otimes}$  equipped with the notion of degree from Definition 2.4 becomes a *filtered algebra*.

**2.6. Approximate factorization property for the vertical arrow.** The conditional cumulants which correspond to the vertical arrow in (1.13) will be denoted by  $\kappa^{\otimes}$ .

**Proposition 2.5.** *The vertical arrow from (1.13) has approximate factorization property.*

*Proof.* In [Śni15, Lemma 6.5] it has been proved that the algebra of  $\alpha$ -polynomial functions  $\mathcal{P}$  is a subalgebra of  $\mathcal{R}$  (as a side remark: this implies also that the diagonal and the vertical arrows in (1.13) are well-defined); also  $\mathcal{P}$  and  $\mathcal{R}$  are equipped with the same multiplication. Furthermore, it has been proved there that each  $\alpha$ -polynomial function  $F \in \mathcal{P}$  which —

regarded as an element of  $\mathcal{P}$  — is of degree at most  $d$  was proved there also to be — this time regarded as an element of  $\mathcal{R}$  — of degree at most  $d$ .

This fact implies that it is enough to show a stronger result that the map (2.2) has approximate factorization property. The remaining part of this section will be devoted to the proof of the latter result; it is stated as Proposition 2.8.  $\square$

2.6.1. *Closed formula for the cumulants  $\kappa^\otimes$ .* Since this does not lead to confusion, the cumulants for the vertical arrow in (1.13) and for the map (2.2) will be denoted by the same symbol  $\kappa^\otimes$ .

Our goal in this section will be to find a closed formula for the cumulant  $\kappa^\otimes(x_1, \dots, x_n)$  for  $x_1, \dots, x_n \in \mathcal{R}$ . By linearity of cumulants we may assume that for each value of the index  $i$ , the function  $x_i \in \mathcal{R}$  has the form

$$(2.4) \quad x_i(\lambda) = \sum_{j_1^{(i)} < \dots < j_{m(i)}^{(i)}} g_i \left( \lambda_{j_1^{(i)}}, \dots, \lambda_{j_{m(i)}^{(i)}} \right).$$

It follows that the pointwise product of functions is given by

$$(2.5) \quad (x_1 \cdots x_n)(\lambda) = \sum_{j_1^{(1)} < \dots < j_{m(1)}^{(1)}} \cdots \sum_{j_1^{(n)} < \dots < j_{m(n)}^{(n)}} \prod_{1 \leq i \leq n} g_i \left( \lambda_{j_1^{(i)}}, \dots, \lambda_{j_{m(i)}^{(i)}} \right).$$

Let us fix some summand on the right-hand side. We denote

$$(2.6) \quad J^{(i)} := \{j_1^{(i)}, \dots, j_{m(i)}^{(i)}\}$$

and consider the graph  $\mathcal{G}$  with the vertex set  $[n] = \{1, 2, \dots, n\}$  the elements of which correspond to the factors; we draw an edge between the vertices  $a$  and  $b$  if the sets  $J^{(a)}$  and  $J^{(b)}$  are not disjoint. The connected components of the graph  $\mathcal{G}$  define a certain partition of the set  $[n]$ .

It follows that the right-hand side of (2.5) can be written in the form

$$(2.7) \quad x_1 \cdots x_n = \sum_{\nu} \prod_{b \in \nu} \widetilde{\kappa^\otimes}(x_i : i \in b),$$

where the sum runs over all set-partitions  $\pi$  of the set  $[n]$  and the product runs over the blocks of  $\pi$ . In the above formula  $\widetilde{\kappa^\otimes}$  denotes the contribution of a prescribed connected component of the graph  $\mathcal{G}$ , i.e.

$$(2.8) \quad \left( \widetilde{\kappa^\otimes}(x_{i_1}, \dots, x_{i_l}) \right) (\lambda_1, \lambda_2, \dots) := \sum_{j_1^{(i_1)} < \dots < j_{m(i_1)}^{(i_1)}} \cdots \sum_{j_1^{(i_l)} < \dots < j_{m(i_l)}^{(i_l)}} \prod_{1 \leq k \leq l} g_{i_k} \left( \lambda_{j_1^{(i_k)}}, \dots, \lambda_{j_{m(i_k)}^{(i_k)}} \right)$$

is defined as the sum over such choices of the indices that the restriction of the above graph  $\mathcal{G}$  to the vertex set  $\{i_1, \dots, i_l\}$  is a connected graph.

The moment-cumulant formula (0.5) takes the following concrete form in our current setup:

$$(2.9) \quad x_1 \cdots x_n = \sum_{\nu} \prod_{b \in \nu} \kappa^{\otimes}(x_i : i \in b).$$

Comparison of (2.7) with (2.9) shows that the quantities  $\widetilde{\kappa^{\otimes}}$  fulfill the same recurrence relations as the cumulants  $\kappa^{\otimes}$ . Since the system of equations (2.9) has the unique solution, it follows that

$$\kappa^{\otimes} = \widetilde{\kappa^{\otimes}}$$

thus (2.8) gives an explicit formula for the latter cumulants.

In this way we proved the following result.

**Lemma 2.6.** *If  $x_i \in \mathcal{R}$  are given by (2.4) then the corresponding cumulant  $\kappa^{\otimes}(x_{i_1}, \dots, x_{i_r})$  is given by the right-hand side of (2.8).*

In the following lemma we shall use the notations from the above proof. Also, for a graph  $\mathcal{G}$  we denote by  $c(\mathcal{G})$  the number of its connected components.

**Lemma 2.7.** *Let a family  $J^{(1)}, \dots, J^{(n)}$  of sets (2.6) be given.*

(1) *Assume that  $\mathcal{G}'$  is a subgraph of  $\mathcal{G}$  with the same vertex set  $[n]$ . Then*

$$\sum_C \left| \bigcup_{a \in C} J^{(a)} \right| \leq m(1) + \cdots + m(n) + c(\mathcal{G}') - n,$$

*where the first sum on the left-hand side runs over the connected components of  $\mathcal{G}'$ .*

(2)

$$\left| \bigcup_{1 \leq a \leq n} J^{(a)} \right| \leq m(1) + \cdots + m(n) + c(\mathcal{G}) - n$$

*Proof.* The proof of the first part of the lemma is a simple induction with respect to the number of the edges of the graph  $\mathcal{G}'$  based on the inclusion-exclusion principle  $|A \cup B| = |A| + |B| - |A \cap B|$ .

The second part of the lemma follows from the first time by setting  $\mathcal{G}' := \mathcal{G}$ .  $\square$

### 2.6.2. The end of the proof of Proposition 2.5.

**Proposition 2.8.** *The map (2.2) has approximate factorization property.*

*Proof.* Let  $x_1, \dots, x_n \in \mathcal{R}$  be of the form (2.4).

Lemma 2.6 gives explicitly the convolution kernel  $(f_r)$  for the cumulant

$$\kappa^\otimes(x_1, \dots, x_n) = \sum_{r \geq 0} \sum_{i_1 < \dots < i_r} f_r(\lambda_{i_1}, \dots, \lambda_{i_r}).$$

More specifically, the summand on the right-hand side for some specified value of  $r$  corresponds to the summands on the right-hand side of (2.8) for which

$$|J^{(1)} \cup \dots \cup J^{(n)}| = r.$$

We keep notations from (2.4). Assume that  $x_i \in \mathcal{R}$  is of degree at most  $d_i$ ; in other words we assume that the corresponding convolution kernel  $g_i$  takes only values in Laurent polynomials of degree at most  $d_i - 2m(i)$ . It follows that the function  $f_r$  takes values in Laurent polynomials of degree at most

$$\sum_{1 \leq i \leq n} d_i - 2m(i).$$

On the other hand, the second part of Lemma 2.7 shows that non-zero contribution can be obtained only for the values of  $r$  which fulfill the bound

$$r = |J^{(1)} \cup \dots \cup J^{(n)}| \leq m(1) + \dots + m(n) + 1 - n.$$

It follows that  $\kappa^\otimes(x_1, \dots, x_n)$  is a row-function of degree at most

$$\left( \sum_{1 \leq i \leq n} d_i - 2m(i) \right) + 2r \leq \left( \sum_{1 \leq i \leq n} d_i \right) + 2(n - 1),$$

which concludes the proof.  $\square$

This completes the proof of Proposition 2.5.

**2.7. Cumulants in terms of moments.** From both definitions of the cumulants (i.e., from (0.4) and (0.5)) it is easy to show that the cumulant

$$\kappa(X_1, \dots, X_n)$$

is a linear combination (with rational coefficients) of the products of the moments of the form

$$(2.10) \quad \prod_{b \in \nu} \mathbb{E} \left( \prod_{i \in b} X_i \right)$$

over set-partitions of the set  $[n]$ .

We will show now that if  $n \geq 2$  then the sum of these coefficients is equal to zero. Indeed, if we set  $X_1 = \dots = X_n = 1$  to be a deterministic

random variable then from both definitions of the cumulants it follows that the corresponding cumulant  $\kappa(1, \dots, 1) = 0$  while each product (2.10) is equal to 1.

**Lemma 2.9.** *For any partitions  $\pi_1, \dots, \pi_l \geq 1$  the function*

$$(2.11) \quad \kappa_{\bullet}^{\otimes}(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l})$$

*is a linear combination (with integer coefficients) of expressions of the form*

$$\bigotimes_{b \in \nu} \text{Ch}_{\prod_{i \in b} \pi_i}$$

*over set-partitions  $\nu$  of the set  $[l]$ . For example, in the case  $l = 3$  the function (2.11) is a linear combination of the following five expressions:*

$$\begin{aligned} \text{Ch}_{\pi_1} \otimes \text{Ch}_{\pi_2} \otimes \text{Ch}_{\pi_3} \quad & \text{Ch}_{\pi_1 \pi_2} \otimes \text{Ch}_{\pi_3} \quad & \text{Ch}_{\pi_1 \pi_3} \otimes \text{Ch}_{\pi_2} \\ & \text{Ch}_{\pi_2 \pi_3} \otimes \text{Ch}_{\pi_1} \quad & \text{Ch}_{\pi_1 \pi_2 \pi_3}. \end{aligned}$$

*Furthermore, if  $l \geq 2$  then the sum of the coefficients in this linear combination is equal to 0.*

*The above results hold true also for the cumulants  $\kappa_{\bullet}$  if the product  $\otimes$  is replaced by the usual pointwise multiplication of the functions.*

## 2.8. Conditional cumulants for the diagonal arrow.

**Lemma 2.10.** *For any partitions  $\pi_1, \dots, \pi_n$*

$$\kappa_{\bullet}^{\otimes}(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_n})(\lambda) = 0$$

*holds true for any Young diagram  $\lambda$  such that  $|\lambda| < |\pi_1| + \dots + |\pi_n|$ .*

*Proof.* We start by showing that Property 1.13 is indeed fulfilled if  $F, G \in \mathcal{R}$  are row functions and  $\otimes$  is the disjoint product.

Assume that  $F$  is a row function with the convolution kernel  $(f_r)$ , see (2.3) and that  $F$  fulfills the assumption (1.11) from Property 1.13. The right-hand side of (2.3) involves only the values of  $f_r$  over  $r \leq \ell(\lambda)$  thus the collection of equalities (2.3) can be viewed as an upper-triangular system of linear equations. It follows immediately that

$$f_r(x_1, \dots, x_r) = 0$$

holds true for all  $r \geq 0$  and all non-negative integers  $x_1, \dots, x_r$  such that

$$x_1 + \dots + x_r < a.$$

The property is fulfilled by the convolution kernel of the row function  $G$  (with the variable  $a$  replaced by  $b$ ).

From the very definition of the disjoint product it follows that the convolution kernel of  $F \otimes G$  also fulfills this property (with the variable  $a$  replaced by  $a + b$ ). This concludes the proof of Property 1.13 for row functions.

Lemma 2.10 is now a straightforward consequence of Lemma 2.9 and Property 1.13.  $\square$

**2.9. Conditional cumulants for a commutative diagram.** We state the following lemma with the commutative diagram (1.13) in mind.

**Lemma 2.11** ([Bri69]). *Assume that  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are commutative unital algebras and let  $\mathbb{E}_{\mathcal{A}}^{\mathcal{B}}$ ,  $\mathbb{E}_{\mathcal{B}}^{\mathcal{C}}$  and  $\mathbb{E}_{\mathcal{A}}^{\mathcal{C}}$  be unital maps between them such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathbb{E}_{\mathcal{A}}^{\mathcal{B}}} & \mathcal{B} \\ & \searrow \mathbb{E}_{\mathcal{A}}^{\mathcal{C}} & \downarrow \mathbb{E}_{\mathcal{B}}^{\mathcal{C}} \\ & & \mathcal{C} \end{array}$$

Then

$$\kappa_{\mathcal{A}}^{\mathcal{C}}(x_1, \dots, x_n) = \sum_{\nu \in \mathcal{P}_n} \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_i : i \in b) : b \in \nu\right).$$

*Example 2.12.*

$$\begin{aligned} \kappa_{\mathcal{A}}^{\mathcal{C}}(x_1) &= \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1)\right), \\ \kappa_{\mathcal{A}}^{\mathcal{C}}(x_1, x_2) &= \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, x_2)\right) + \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_2)\right) \\ \kappa_{\mathcal{A}}^{\mathcal{C}}(x_1, x_2, x_3) &= \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, x_2, x_3)\right) + \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_2, x_3)\right) + \\ &\quad \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_2), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, x_3)\right) + \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_3), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_1, x_2)\right) + \\ &\quad \kappa_{\mathcal{B}}^{\mathcal{C}}\left(\kappa_{\mathcal{A}}^{\mathcal{B}}(x_1), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_2), \kappa_{\mathcal{A}}^{\mathcal{B}}(x_3)\right). \end{aligned}$$

**2.10. Condition (Z3) for row functions of small degree.**

**Lemma 2.13** ([Śni15, Lemma 6.8]). *Let  $d \geq 1$  be an integer and assume that  $F \in \mathcal{R}$  is of degree at most  $d - 1$ .*

*Then for each integer  $k \geq 0$  and each Young diagram  $\lambda = (\lambda_1, \lambda_2, \dots)$*

$$[A^{d-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^{\text{sym}}(\lambda_1, \lambda_2, \dots) = 0.$$

**2.11. Proof of the main result.** We are now ready to show the proof of Theorem 1.9. For Reader's convenience we will restate this theorem in the following form.

**Theorem 2.14** (Reformulation of Theorem 1.9). *For any partitions  $\pi_1, \dots, \pi_l \in \mathcal{P}$  the corresponding conditional cumulant*

$$F := \kappa_{\bullet}(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}) \in \mathcal{P}$$

is of degree at most

$$|\pi_1| + \cdots + |\pi_l| + \ell(\pi_1) + \cdots + \ell(\pi_l) - 2(l-1).$$

*Proof.* We use induction over  $l$ . For the induction base  $l = 1$

$$F = \kappa_\bullet(\text{Ch}_{\pi_1}) = \text{Ch}_{\pi_1}$$

and there is nothing to prove. In the following we shall consider the case  $l \geq 2$ ; we assume that the statement of the theorem holds true for all  $l' < l$ .

We start with an observation that if for some value of the index  $i$  we have  $\pi_i = \emptyset$  then  $\text{Ch}_{\pi_i} = 1$  is the unit in  $\mathcal{P}$  thus (for  $l \geq 2$ ) the corresponding cumulant vanishes:

$$F = \kappa_\bullet(\text{Ch}_{\pi_1}, \dots, 1, \dots, \text{Ch}_{\pi_l}) = 0$$

and the claim holds trivially true. From the following on we shall assume that  $\pi_1, \dots, \pi_l \neq \emptyset$  are all non-empty.

We denote

$$d := |\pi_1| + \cdots + |\pi_l| + \ell(\pi_1) + \cdots + \ell(\pi_l).$$

We will use a nested induction over  $j \in \{0, \dots, l-1\}$  and show that  $F$  is of degree at most  $d - 2j$ . This result (for the special choice  $j = l-1$ ) would finish the proof of the inductive step with respect to the variable  $l$  and thus would conclude the proof.

Lemma 2.9 (in the alternative formulation, for the cumulants  $\kappa^\bullet$ ) implies that  $F \in \mathcal{P}$  is of degree at most  $d$  and thus the induction base  $j = 0$  holds trivially true.

The inductive hypothesis with respect to the variable  $j$  states that  $F$  is of degree (at most)  $d - 2j$  for some choice of  $j \in \{0, \dots, l-2\}$ . We shall prove the induction step with respect to the variable  $j$  by applying Lemma 2.3 twice.

*The first application of Lemma 2.3.* Our strategy is to apply Lemma 2.3 either:

- in the original formulation (in the case  $j = 0$ ), or,
- in the alternative formulation (in the case  $j \geq 1$ )

for

$$n := |\pi_1| + \cdots + |\pi_l| - j - 2 \geq 2,$$

$$r := \ell(\pi_1) + \cdots + \ell(\pi_l) - j \geq 2;$$

we first check that its assumptions are fulfilled.

*Assumption (Z1).* This assumption is just the inductive hypothesis.



*Assumption (Z2).* We have to verify this assumption only in the case  $j = 0$ . By Lemma 2.9 and condition (K2) from Definition 1.5 it follows that

$$(2.12) \quad \mathbb{Y} \ni (\lambda_1, \dots, \lambda_m) \mapsto F(\lambda_1, \dots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]$$

is a priori a polynomial of degree  $|\pi_1| + \dots + |\pi_l|$  and its homogeneous top-degree part is equal to some multiple of

$$A^{|\pi_1| + \dots + |\pi_l| - \ell(\pi_1) - \dots - \ell(\pi_l)} p_{\pi_1 \dots \pi_l}(\lambda_1, \dots, \lambda_m).$$

However, since  $l \geq 2$ , the second part of Lemma 2.9 implies that this multiple is actually equal to zero. In other words, (2.12) is a polynomial of degree at most  $|\pi_1| + \dots + |\pi_l| - 1 = n - 1$ , as required.

*Assumption (Z4).* Lemma 2.9 and condition (K4) from Definition 1.5 imply that for any Young diagram  $\lambda$  the evaluation  $F(\lambda)$  is a Laurent polynomial of degree at most  $|\pi_1| + \dots + |\pi_l| - \ell(\pi_1) - \dots - \ell(\pi_l) = n - r$  as required.

*Assumptions (Z3) and (Z3a).* Since  $\kappa^\bullet(x) = x$  it follows that

$$(2.13) \quad F = \kappa_\bullet(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}) = \kappa^\bullet(\kappa_\bullet(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l})) = \kappa_\bullet^\otimes(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}) - \sum_{\nu \neq 1} \kappa^\bullet(\kappa_\bullet(\text{Ch}_{\pi_i} : i \in b) : b \in \nu),$$

where the last equality follows from Lemma 2.11; the sum on the right-hand side runs over set-partitions of  $[l]$  which are different from the maximal partition  $1 = \{\{1, \dots, l\}\}$ . We will substitute the right-hand side into (2.1) and we will investigate the resulting expression.

Firstly, Lemma 2.10 shows that  $(\kappa_\bullet^\otimes(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}))(\lambda) = 0$  for all  $\lambda \in \mathbb{Y}$  such that  $|\lambda| \leq |\pi_1| + \dots + |\pi_l| - 1$  thus for any integer  $k \geq r$

$$(2.14) \quad \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} (\kappa_\bullet^\otimes(\text{Ch}_{\pi_1}, \dots, \text{Ch}_{\pi_l}))^{\text{sym}}(\lambda) = 0$$

for all  $\lambda \in \mathbb{Y}$  such that

$$|\lambda| \leq |\pi_1| + \dots + |\pi_l| - 1 - k = (n + r - 2k) + (j - 1) + (k - r).$$

It follows that:

- if  $j = 0$  then (2.14) holds true for all values of  $k$  and  $\lambda$  which appear in condition (Z3);
- if  $j \in \{1, \dots, l - 2\}$  then (2.14) holds true for all  $k$  and  $\lambda$  appear in condition (Z3a).

Secondly, let us fix the value of the set-partition  $\nu \neq 1$  and let us investigate the corresponding summand on the right-hand side of (2.13). From the

inductive hypothesis over the variable  $l$  it follows for each block  $b \in \nu$  that the cumulant  $\kappa_\bullet(\text{Ch}_{\pi_i} : i \in b) \in \mathcal{P}$  is of degree at most

$$\left( \sum_{i \in b} |\pi_i| + \ell(\pi) \right) - 2(|b| - 1).$$

Thus Proposition 2.5 implies that

$$G := \kappa^\bullet \left( \kappa_\bullet(\text{Ch}_{m_i} : i \in b) : b \in \nu \right) \in \mathcal{R}$$

is of degree (at most)

$$\sum_{b \in \nu} \left[ \left( \sum_{i \in b} |\pi_i| + \ell(\pi) \right) - 2(|b| - 1) \right] + 2 - 2|\nu| =$$

$$d - 2(l - 1) \leq n + r - 2.$$

The latter bound on the degree of  $G$  and Lemma 2.13 imply that

$$[A^{n+r-2k}] \Delta_{\lambda_1} \dots \Delta_{\lambda_k} G(\lambda_1, \dots, \lambda_i) = 0.$$

It follows that:

- if  $j = 0$  then condition (Z3) holds true;
- if  $j \in \{1, \dots, l - 2\}$  then stronger condition (Z3a) holds true.

*Conclusion of the first application of Lemma 2.3.* Lemma 2.3 implies that  $F$  is of degree at most  $d - 2j - 1$ .

*The second application of Lemma 2.3.* We shall apply Lemma 2.3 again; this time in the alternative formulation for

$$n := |\pi_1| + \dots + |\pi_l| - j - 2 \geq 2,$$

$$r' := r - 1 = \ell(\pi_1) + \dots + \ell(\pi_l) - j - 1 \geq 1.$$

Verification that the assumptions are fulfilled is not very difficult: firstly, we do not have to verify assumption (Z2)). Secondly, for the remaining conditions it is enough to revisit the above first application of Lemma 2.3 and make some minor adjustments. Indeed, verification of the assumption (Z4) is easier now because  $r' < r$  while the verification of (Z3a) remains essentially the same as before.

Thus Lemma 2.3 shows that  $F$  is of degree at most  $d - 2j - 2$ . This concludes the proof of the induction step over variable  $j$ .  $\square$

## ACKNOWLEDGMENTS

All numerical examples presented in the current paper are based on data provided by Michel Lassalle [Las09b].

I thank Maciej Dołęga and Valentin Féray for several years of collaboration on topics related to the current paper.

Research supported by *Narodowe Centrum Nauki*, grant number 2014/15/B/ST1/00064.

## REFERENCES

- [Bia03] Philippe Biane. Characters of symmetric groups and free cumulants. In *Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001)*, volume 1815 of *Lecture Notes in Math.*, pages 185–200. Springer, Berlin, 2003.
- [BO05] Alexei Borodin and Grigori Olshanski. Z-measures on partitions and their scaling limits. *European Journal of Combinatorics*, 26(6):795–834, 2005.
- [Bri69] David R. Brillinger. The calculation of cumulants via conditioning. *Annals of the Institute of Statistical Mathematics*, 21(1):215–218, 1969.
- [Bur16] Adam Burchardt. Some structure constants for Jack polynomials. In preparation, 2016.
- [DF16] Maciej Dołęga and Valentin Féray. Gaussian fluctuations of Young diagrams and structure constants of Jack characters. *Duke Math. J.*, to appear, doi:10.1215/00127094-3449566; also available as preprint arXiv:1402.4615v3, 2016.
- [DFŚ10] Maciej Dołęga, Valentin Féray, and Piotr Śniady. Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations. *Adv. Math.*, 225(1):81–120, 2010.
- [DFŚ14] Maciej Dołęga, Valentin Féray, and Piotr Śniady. Jack polynomials and orientability generating series of maps. *Sém. Lothar. Combin.*, 70:Art. B70j, 50, 2014.
- [DH92] Persi Diaconis and Phil Hanlon. Eigen-analysis for some examples of the metropolis algorithm. *Contemporary Mathematics*, 138:99–117, 1992.
- [DŚ16] Maciej Dołęga and Piotr Śniady. Gaussian fluctuations of Jack-deformed random Young diagrams. Work in progress, 2016.
- [Fér09] Valentin Féray. Combinatorial interpretation and positivity of Kerov’s character polynomials. *J. Algebraic Combin.*, 29(4):473–507, 2009.
- [Fis28] R. A. Fisher. Moments and Product Moments of Sampling Distributions. *Proc. London Math. Soc.*, S2-30(1):199, 1928.
- [FŚ11] Valentin Féray and Piotr Śniady. Asymptotics of characters of symmetric groups related to Stanley character formula. *Ann. of Math. (2)*, 173(2):887–906, 2011.
- [GJ96] I. P. Goulden and D. M. Jackson. Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. *Trans. Amer. Math. Soc.*, 348(3):873–892, 1996.
- [GR05] Adriano Garsia and Jeffrey B. Remmel. Breakthroughs in the theory of Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 102(11):3891–3894 (electronic), 2005.

- [Hal81] A. Hald. T. N. Thiele's contributions to statistics. *Internat. Statist. Rev.*, 49(1):1–20 (one plate), 1981.
- [IK99] V. Ivanov and S. Kerov. The algebra of conjugacy classes in symmetric groups, and partial permutations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 256(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 3):95–120, 265, 1999.
- [Jac71] Henry Jack. A class of symmetric polynomials with a parameter. *Proc. Roy. Soc. Edinburgh Sect. A*, 69:1–18, 1970/1971.
- [Kan93] Jyoichi Kaneko. Selberg integrals and hypergeometric functions associated with jack polynomials. *SIAM journal on mathematical analysis*, 24(4):1086–1110, 1993.
- [KO94] Serguei Kerov and Grigori Olshanski. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(2):121–126, 1994.
- [Las08] Michel Lassalle. A positivity conjecture for Jack polynomials. *Math. Res. Lett.*, 15(4):661–681, 2008.
- [Las09a] Michel Lassalle. Jack polynomials and free cumulants. *Adv. Math.*, 222(6):2227–2269, 2009.
- [Las09b] Michel Lassalle. Jack polynomials and free cumulants. <http://igm.univ-mlv.fr/~lassalle/free.html>, 2009. Accessed: 1/12/2012.
- [Oko03] A. Okounkov. The uses of random partitions. In *Fourteenth International Congress on Mathematical Physics*, pages 379–403. World Scientists, 2003.
- [OO97] Andrei Okounkov and Grigori Olshanski. Shifted Jack polynomials, binomial formula, and applications. *Math. Res. Lett.*, 4(1):69–78, 1997.
- [RŚ08] Amarpreet Rattan and Piotr Śniady. Upper bound on the characters of the symmetric groups for balanced Young diagrams and a generalized Frobenius formula. *Adv. Math.*, 218(3):673–695, 2008.
- [Śni06a] Piotr Śniady. Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Related Fields*, 136(2):263–297, 2006.
- [Śni06b] Piotr Śniady. Gaussian fluctuations of representations of wreath products. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 9(4):529–546, 2006.
- [Śni15] Piotr Śniady. Top degree of Jack characters. Preprint arXiv:1506.06361v2, 2015.
- [Sta06] Richard P. Stanley. A conjectured combinatorial interpretation of the normalized irreducible character values of the symmetric group. Preprint arXiv:math.CO/0606467, 2006.

WYDZIAŁ MATEMATYKI I INFORMATYKI, UNIWERSYTET IM. ADAMA MICKIEWICZA,  
COLLEGIUM MATHEMATICUM, UMULTOWSKA 87, 61-614 POZNAŃ, POLAND,

INSTYTUT MATEMATYCZNY, POLSKA AKADEMIA NAUK, UL. ŚNIADECKICH 8,  
00-956 WARSZAWA, POLAND

*E-mail address:* piotr.sniady@amu.edu.pl